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RADIATION IN A DIFFUSING MEDIUM  
WITH APPLICATION TO SNOW

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**RADIATION IN A DIFFUSING MEDIUM  
WITH APPLICATION TO SNOW**

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## FOREWORD

The enclosed analysis is original as conceived, but is not original to the field fundamentally. The first reference in which the basic differential equations were established was that by A. Schuster (Astrophysical Journal, vol. 21, 1, (No. 5)) for the radiation through gaseous stars. Further elaboration of this method was performed by L. Silberstein (Philosophical Magazine, vol. 4, 129 (1927)), J. W. Ryde (Proceedings of the Royal Society of London, vol. 131A, 451, (1930)) and S. Q. Duntley (Journal of the Optical Society of America, vol. 32, 61 (1942)). Each succeeding author refined and extended the basic precepts until the analysis by Duntley contained eight constants describing various physical properties of the medium. Notwithstanding these facts, the application of the method has furnished an insight into correlating parameters for reflection and transmission of a snow cover. In addition, the transmission was shown to be related to the albedo of the cover and to be attenuated in an exponential manner.

## RADIATION IN A DIFFUSING MEDIUM WITH APPLICATION TO SNOW

### Introduction

The transmission and reflection characteristics of multiparticle media such as snow are normally presumed to be independent for purposes of correlating experimental measurements. If it is postulated that the medium is composed of homogeneous semi-infinite slabs of finite incremental thickness, the reflection and transmission of,  $n$ , unit thicknesses may be analyzed by the procedures described by Benford (1946). This method is satisfactory if only a few slabs are considered but as the number of slabs becomes very large, calculation of the transmission is difficult. In addition, the analysis presented by Benford is not satisfactory for determination of the transmission at various depths within the medium. Furthermore, no method of correlation of experimental measurements is indicated by this analysis.

In view of these difficulties, an attempt was made to approach the problem in a different manner. The basic assumptions of the analysis are:

- (a) The medium is composed of a perfectly diffusing material with a uniform distributed reflectivity,  $r$ , which is defined as the reflectivity per unit distance, or the number of reflecting surfaces per unit distance times a constant reflectivity per surface.
- (b) The absorption coefficient,  $k$ , is assumed uniform.
- (c) Emission within the medium is negligible, i.e., the temperature is low compared to the energy sources.

### Analysis

The total radiation downwards at any depth,  $x$ , is given the symbol,  $Y$ , BTU/hrft<sup>2</sup>, and the radiation upward is designated by,  $Z$ , BTU/hrft<sup>2</sup>.

The energy passing downward through the medium, i.e., positive  $x$  direction, in a distance,  $dx$ , is decreased by absorption and reflection and increased by

reflection of the upward component:

$$dY = -k Y dx - r Y dx + r Z dx \quad (1)$$

Similarly, the change in the energy passing upward is:

$$dZ = k Z dx - r Y dx + r Z dx \quad (2)$$

An illustrative diagram of the energy transfer is shown in Figure 1 below:

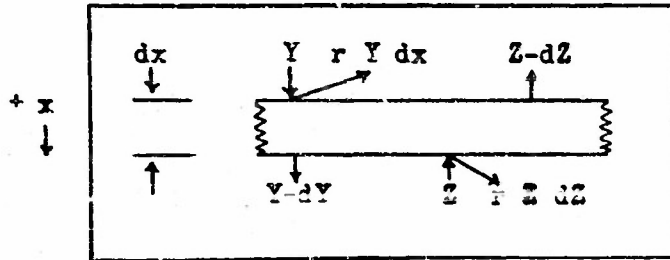


Figure 1.

Division of equation (1) by (2) eliminates  $dx$  and  $Y$  may be found in terms of  $Z$  by integration, giving:

$$(k + r) ZY - \frac{r}{2} Y^2 - \frac{r}{2} Z^2 = 0 \quad (3)$$

The value of the integration constant is to be evaluated from the boundary conditions after obtaining  $Y$  and  $Z$  in terms of  $k$  and  $r_0$ . Solving for  $Z$ , one obtains

$$Z = Y \left[ \frac{k + r}{r} \pm \sqrt{\left(\frac{k + r}{r}\right)^2 - 1} \right] \quad (4)$$

Substitution of (4) into (1) and integrating gives

$$Y = y_1 e^{-x \sqrt{(k+r)^2 - r^2}} + y_2 e^{x \sqrt{(k+r)^2 - r^2}}$$

or

$$Y = y_1 e^{-\beta x} + y_2 e^{\beta x} \quad (5)$$

where

$$\beta = \sqrt{(k + r)^2 - r^2} \quad (6)$$

From equations (4) and (5), the expression obtained for  $Z$  is:

$$Z = \left[ \frac{k + r - \beta}{r} \right] y_1 e^{-\beta x} + \left[ \frac{k + r + \beta}{r} \right] y_2 e^{\beta x} \quad (7)$$

The boundary conditions are (see Figure 2) at  $x = c$ :

$$Y_0 = (1 - r_0) Y_1 + r_0 Z_0 \quad (8)$$

and at  $x = b$ :

$$Z_b = (1 - r_b) Z_1 + r_b Y_b \quad (9)$$

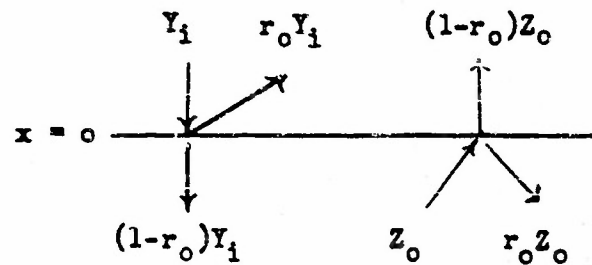
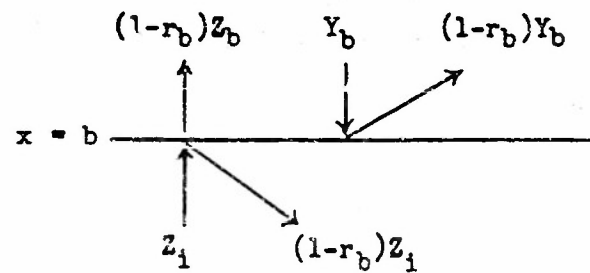


Figure 2.



The boundary conditions give

$$x = 0: y_1 + y_2 = (1 - r_0) Y_1 + r_0 Z_0$$

$$Z_0 = \left( \frac{k + r - \beta}{r} \right) y_1 + \left( \frac{k + r + \beta}{r} \right) y_2$$

or

$$(1 - r_o)Y_1 = \left[ 1 - \frac{r_o(k + r - \beta)}{r} \right] y_1 + \left[ 1 - \frac{r_o(k + r + \beta)}{r} \right] y_2 \quad (10)$$

$$x = b: Y_b = \left[ y_1 e^{-\beta b} + y_2 e^{+\beta b} \right] r_b$$

$$Z_b = \left[ \frac{k + r - \beta}{r} \right] y_1 e^{-\beta b} + \left[ \frac{k + r + \beta}{r} \right] y_2 e^{+\beta b}$$

or

$$(1 - r_b)Z_1 = y_1 \left[ \frac{k + r - \beta}{r} - r_b \right] e^{-\beta b} + y_2 \left[ \frac{k + r + \beta}{r} - r_b \right] e^{+\beta b} \quad (11)$$

From equations (11) and (12)

$$y_1 = \frac{(1 - r_o) \left[ \frac{k + r + \beta}{r} - r_b \right] e^{+\beta b} Y_1 - (1 - r_b) \left[ 1 - r_o \frac{(k + r + \beta)}{r} \right] Z_1}{D_1 e^{+\beta b} - D_2 e^{-\beta b}} \quad (12)$$

$$y_2 = \frac{(1 - r_b) \left[ 1 - \frac{r_o(k + r - \beta)}{r} \right] Z_1 - \left[ \frac{k + r - \beta}{r} - r_b \right] (1 - r_o) e^{-\beta b} Y_1}{D_1 e^{+\beta b} - D_2 e^{-\beta b}} \quad (13)$$

where

$$D_1 = \left[ \frac{k + r + \beta}{r} - r_b \right] \left[ 1 - \frac{r_o(k + r - \beta)}{r} \right] \quad (14)$$

$$D_2 = \left[ \frac{k + r - \beta}{r} - r_b \right] \left[ 1 - \frac{r_o(k + r + \beta)}{r} \right] \quad (15)$$

Equations (12) through (15) and equations (5) and (7) determine the functions Y and Z. The majority of applications in practice would have  $Z_1 = 0$ . For this condition:

$$Y = \frac{(1-r_0) Y_1}{D_1 e^{\beta b} - D_2 e^{-\beta b}} \left\{ \left[ \frac{k+r+\beta}{r} - r_b \right] e^{\beta b} e^{-\beta x} - \left[ \frac{k+r-\beta}{r} - r_b \right] e^{-\beta b} e^{\beta x} \right\} \quad (16)$$

$$Z = \frac{(1-r_0) Y_1}{D_1 e^{\beta b} - D_2 e^{-\beta b}} \left\{ \left[ \frac{k+r-\beta}{r} \right] \left[ \frac{k+r+\beta}{r} - r_b \right] e^{\beta b} e^{-\beta x} - \left[ \frac{k+r+\beta}{r} \right] \left[ \frac{k+r-\beta}{r} - r_b \right] e^{-\beta b} e^{\beta x} \right\} \quad (17)$$

For the case of a thick layer, e.g. deep snow cover, equations (16) and (17) reduce to ( $b \rightarrow \infty$ ):

$$Y = \frac{(1-r_0) Y_1 e^{-\beta x}}{\left[ 1 - \frac{r_0(k+r-\beta)}{r} \right]} \quad (18)$$

$$Z = \frac{\left( \frac{k+r-\beta}{r} \right) (1-r_0) Y_1 e^{-\beta x}}{\left[ 1 - \frac{r_0(k+r-\beta)}{r} \right]} \quad (19)$$

The value of the reflectance,  $R_0$  for a diffuse medium, is seen to be (refer to Figure 2):

$$R_0 = \frac{r_0 Y_1 + (1-r_0) Z_0}{Y_1} \quad (20)$$

The transmission factor may be defined as the net energy passing a plane at distance  $x$ :

$$T_x = \frac{Y_x - Z_x}{Y_1} \quad (21)$$

While the foregoing equations were developed for a distributed reflectivity, they can be applied with small error to a series of finite absorbing slabs. The distributed reflectivity in this case is taken as the number of



reflections per unit distance times the reflectivity or:

$$r = \frac{2(n-1)r_s}{\Delta x} \quad (22)$$

where,  $n$ , is the number of slabs of thickness,  $\Delta x$  and surface reflectivity  $r_s$ .

The calculated reflectances and transmittances from these equations can be compared with Benford's equations as a check upon the error involved in the assumption of a distributed reflectivity.

The results by Benford (1946) gave for transmission and reflection of a finite number of similar slabs:

$$R_{2n} = R_{2n-1} + \frac{(R_{2n-1})(T_{2n-1})^2}{1 - (R_{2n-1})^2} = R_{2n-1} [1 + T_{2n}] \quad (23)$$

$$T_{2n} = \frac{T_{2n-1}^2}{1 - R_{2n-1}^2} \quad (24)$$

For an infinite (large) number of slabs, the expression for  $R_0$  derived by Benford was:

$$R_0 = \frac{(1 + R_1^2 - T_1^2) - \left[ (1 + R_1^2 - T_1^2)^2 - 4R_1^2 \right]^{1/2}}{2R_1} \quad (25)$$

The values of  $R_1$  and  $T_1$  are the reflectivity and transmissivity of a single slab. In terms of  $r_s$  and  $k$  this may be shown to be (see Appendix):

$$R_1 = r_s + \frac{r_s(1-r_s)^2 e^{-2k\Delta x}}{1 - r_s^2 e^{-2k\Delta x}} \quad (26)$$

$$T_1 = \frac{(1-r_s)^2 e^{-k\Delta x}}{1 - r_s^2 e^{-2k\Delta x}} \quad (27)$$

An important fact to be noted is that as  $\Delta x \rightarrow 0$ ,  $R_1$  is not equal to zero.

This is contrary to the hypothesis of Benford and results from the fundamental

concept that reflectivity is a surface property. Also worthy of note is the lack in the work of Benford of a method for determining the transmission at any depth within the slabs, other than a backward reiteration process. Benford presented a method of calculation for this but the basis for the procedure was the postulate that  $R = 0$  and  $T = 1.0$  at zero thickness.

The work of Benford does furnish a means of establishing the validity of the analysis presented herein. The following values of  $k$  and  $r_s$  were selected ( $r_s = r_0$ ):

<u><math>k \text{ in}^{-1}</math></u>	<u><math>r_s</math></u>
1	.01
10	.1
100	.9

and a constant slab thickness of 0.005 inches was selected. The number of slabs selected were 2, 4 and  $\infty$ . The results of the calculations are shown in Table 1. The agreement of the two methods is within 2 percent or less of unity and the major portion of the differences may be attributed to significant figure errors in calculation.

TABLE I

Comparison of results obtained by the analysis  
of Benford and the method presented herein.

$r_s$	Benford Present Analysis				Benford Present Analysis	
	T	R	T	R	$R_o$	$R_o$
	$k = 1.0$ $x = 0.010^n \quad n = 2$				$k = 1.0$	
0.01	.952	.038	.952	.041	.502	.503
0.10	.685	.305	.683	.291	.815	.804
0.90	.024	.967	.045	.945	.977	.958
	$k = 10$				$k = 10$	
0.01	.869	.035	.875	.025	.147	.153
0.10	.623	.282	.632	.273	.519	.526
0.90	.009	.939	.022	.925	.940	.931
	$k = 100$				$k = 100$	
0.01	.353	.018	.355	.020	.021	.029
0.10	.247	.162	.251	.170	.173	.220
0.90	.0004	.905	.0023	.907	.905	.908
	$k = 1$ $x = 0.020^n \quad n = 4$					
0.01	.906	.073	.907	.073		
0.10	.517	.463	.536	.444		
0.90	.009	.976	.033	.951		
	$k = 10$					
0.01	.757	.062	.759	.062		
0.10	.422	.402	.437	.388		
0.90	0	.940	.013	.925		
	$k = 100$					
0.01	.125	.021	.125	.024		
0.10	.063	.172	.065	.195		
0.90	0	.905	.001	.907		

### Discussion

This analysis was initiated in an attempt to resolve a portion of the difficulties encountered in the measurement and correlation of transmission and reflection of a snow cover. While not specified, it must be remembered that  $r_s$  and  $k$  may vary with wavelength, and in general monochromatic values should be employed and an integration performed for specific sources.

The results obtained considered irradiation of both the top and bottom of the medium, but for a snow cover the primary source of irradiation will be from one side only. The equations allow the substitution of any reflectivity,  $r_b$ , which could be taken as the reflectivity of the ground in case of a snow cover. For a snow cover it has been shown (SIPRE Report No. 4, 1951) that for a depth greater than 12 to 24 inches, dependent upon the type and nature of the snow, that transmission of solar energy is negligible.

Consequently, the results expressed in equations (18) and (19) can be considered pertinent. Substitution of these expressions in equations (20) and (21) yields:

$$R_o = A = r_o + \frac{\left(\frac{k+r-\beta}{r}\right)(1-r_o)^2}{\left[1 - r_o\left(\frac{k+r-\beta}{r}\right)\right]} \quad (26)$$

and

$$T_x = \frac{(1-r_o) e^{-\beta x}}{\left[1 - r_o\left(\frac{k+r-\beta}{r}\right)\right]} \left[1 - \frac{k+r-\beta}{r}\right] \quad (29)$$

where

$A$  = albedo

Equation (29) can be written

$$T_x = (1 - A) e^{-\beta x} \quad (30)$$

Thus, the net transmission of the medium is directly related to the albedo. Furthermore, the transmission of a particular snow cover will be

exponential with depth ( $x$ ) and an attenuation coefficient may be obtained.

Wide variance between different observed values of this attenuation coefficient should be expected since  $k$ ,  $r_s$ , and  $r_0$  influence the value obtained.

The number of variables involved may be reduced by assuming,  $k$ , is a constant for all snows. This assumption is reasonably valid as the indices of refraction (directly related to  $k$ ) for ice and water are essentially equal and the presumption that this is true for an ice crystal (snow) follows. Consequently, the primary variables are  $\Delta x$  and the reflectivities  $r_0$  and  $r_s$ . It is seen from equation (23) that the distributed reflectivity is defined for the case of a deep snow ( $n \rightarrow \infty$ ):

$$r = \frac{2r_s}{\Delta x} \quad (31)$$

It is thought that  $\left(\frac{r_s}{\Delta x}\right)$  will depend upon the shape and  $\Delta x$  upon the size of the snow particles.

As a snow ages, physical changes in the crystalline structure occur. Melting, compaction, etc. will alter the shape and size of the individual crystals causing changes in the transmission and reflection of the snow cover. In a new snow with small crystals  $\Delta x$  is small and hence  $r$  is large compared to  $k$ , resulting in a high albedo. As the snow ages and crystals grow larger,  $r$  will decrease relative to  $k$  resulting in a decrease in albedo. The limiting case would be a plane layer of homogeneous ice which would have a very low albedo. It must be remembered that this discussion applies only to clean snow.

From the foregoing discussion, the important variables affecting  $T_x$  and  $A$ , have been reduced to:

$r$  - the distributed reflectivity

$r_0$  - the first surface reflectivity

The first,  $r$ , is suggested as a parameter varying between types of snow and as a possible means of correlating the data. The latter variable  $r_0$ , is a surface condition which will also be dependent upon the type of snow but should

have comparatively small effect upon the albedo ( $A$ ) and transmission ( $T_x$ ). The supposition that these may be treated separately is subject to question and is proposed as a first attempt to secure more satisfactory correlation of experimental measurements of snow transmission and reflection. Variations within the cover have been neglected and this may negate efforts to obtain descriptive parameters of various types of snow covers.

### Conclusions

1. The transmission through an idealized snow cover has been shown to be directly related to the albedo and to be an exponentially decreasing function.

2. Two parameters have been proposed as possible correlating factors for transmission and albedo measurements. The first factor is a characteristic of the surface condition of the cover and the second is a characteristic of the snow beneath the surface.

### Experimental Verification

The experimental verification of the applicability of this analysis to snow is contemplated. Measurements will be made in a snow cover utilizing the Solarimeter and Albedometer, described previously (Gier, Dunkle 1953).

Facilities are available locally for such work.

### REFERENCES

Benford, F. (1946) Radiation in a diffusing medium, Journal of the Optical Society of America, vol 36.

University of Minnesota (1951) Review of the Properties of Snow and Ice. SIPRE Report 4. Institute of Technology, EMS, Minneapolis. 156 pp.

Gier, J. T. and R. V. Dunkle (1953) Progress report for the year June 27, 1952 to June 27, 1953, Berkeley: The University of California, Institute of Engineering Research, Series No. 62, Issue No. 1, Contract No. DA-11-190-ENG-3, 73 pages.

NOMENCLATURE

A	= albedo; reflectivity of a medium of large thickness (i.e. $x \rightarrow \infty$ ) defined by equation (28)
e	= natural logarithm base
k	= absorption or extinction coefficient; (inches) <sup>-1</sup>
n	= number of slabs
r	= distributed reflectivity; defined by either equation (22) or (31)
r <sub>b</sub>	= reflectivity of the lower boundary of the medium
r <sub>o</sub>	= reflectivity of the upper boundary of the medium
r <sub>s</sub>	= reflectivity of a single slab or lamina of the medium
R <sub>o</sub>	= total reflectivity of the medium at the upper boundary; defined by equation (20)
R <sub>2n</sub>	= reflectivity of 2 <sup>n</sup> slabs; equation (23)
R <sub>2n-1</sub>	= reflectivity of 2 <sup>n-1</sup> slabs; equation (23) and (24)
R <sub>1</sub>	= reflectivity of one slab; equation (25) and Appendix
T <sub>x</sub>	= effective transmission at depth x; defined by equation (21)
T <sub>2n</sub>	= effective transmission of 2 <sup>n</sup> slabs; equations (23) and (24)
T <sub>1</sub>	= effective transmission of one slab; equation (25) and Appendix
t	= transmission between boundaries of a unit slab; Appendix
x	= thickness; inches
y <sub>1</sub> and y <sub>2</sub>	= constants; defined by equation (4)
Y	= energy downward in medium; BTU/hrft <sup>2</sup>
Y <sub>b</sub>	= energy incident upon lower boundary; BTU/hrft <sup>2</sup>
Y <sub>i</sub>	= energy incident upon the upper surface of the medium at $x = 0$ , BTU/hrft <sup>2</sup>
Y <sub>o</sub>	= energy incident upon upper boundary; BTU/hrft <sup>2</sup>
Z	= energy upward in medium; BTU/hrft <sup>2</sup>
Z <sub>b</sub>	= energy incident upon the lower surface of the medium; BTU/hrft <sup>2</sup>
Z <sub>i</sub>	= energy incident upon the lower surface of the medium at $x = b$ , BTU/hrft <sup>2</sup>

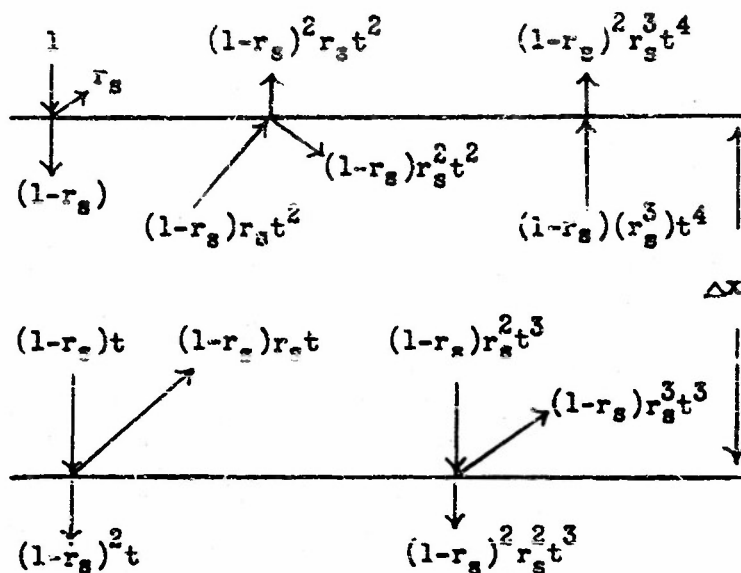
NOMENCLATURE (Cont'd)

- $z_o$  = energy incident upon the upper surface of the medium; BTU/hr-ft<sup>2</sup>
- $\beta$  = variable; defined by equation (7)
- $\Delta$  = increment



APPENDIX

## Transmission and Reflection from a Single Slab



$$R_1 = r_s + (1-r_s)^2 r_s t^2 + (1-r_s)^2 r_s^3 t^4 + (1-r_s)^2 r_s^5 t^6 + \dots$$

$$= r_s + (1-r_s)^2 r_s t^2 \left[ 1 + r_s^2 t^2 + r_s^4 t^4 + \dots \right]$$

$$= r_s \left[ 1 + \frac{(1-r_s)^2 t^2}{1-r_s^2 t^2} \right]$$

$$T_1 = (1-r_s)^2 t + (1-r_s)^2 r_s^2 t^3 + (1-r_s)^2 r_s^4 t^5 + \dots$$

$$= (1-r_s)^2 t \left[ 1 + r_s^2 t^2 + r_s^4 t^4 + \dots \right]$$

$$= \frac{(1-r_s)^2 t}{1-r_s^2 t^2}$$

If  $t = e^{-k\Delta x}$  the above equations become:

$$R_1 = r_s \left[ 1 + \frac{(1-r_s)^2 e^{-2k\Delta x}}{1-r_s^2 e^{-2k\Delta x}} \right]$$

$$T_1 = \frac{(1-r_s)^2 e^{-k\Delta x}}{1-r_s^2 e^{-2k\Delta x}}$$